Implementation of Fuzzy Random Variable in Imprecise Modelling of Contaminant Migration through a Soil Layer

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Abstract - Fuzzy random variables possess several interpretations. Historically, they were proposed either as a tool for handling linguistic label information in statistics or to represent uncertainty about classical random variables. Accordingly, there are two different approaches to the definition of the variance of a fuzzy random variable. In the first one, the variance of the fuzzy random variable is defined as a crisp number that makes it easier to handle in further processing. In the second case, the variance is defined as a fuzzy interval, offering a gradual description of our incomplete knowledge about the variance of an underlying, imprecisely observed, classical random variable. In this work, we first discuss another view of fuzzy random variables that comes down to a set of random variables induced by a fuzzy relation describing an ill-known conditional probability. This view leads to yet another definition of the variance of a fuzzy random variable, in the context of the theory of imprecise probabilities. The new variance is a real interval, which achieves a compromise between both previous definitions in terms of representation simplicity. Our main objective is to demonstrate, with the help of simple examples, the practical significance of these definitions of variance induced by various existing views of fuzzy random variables. Finally, fuzzy random variable concept is implemented in modeling of contaminant migration through a soil layer. The transport of contaminant through a saturated soil layer is modeled by advection, dispersion, sorption and first order degradation. The parameters of the contaminant migration model such as seepage velocity, porosity of soil, dispersion coefficient and distribution coefficient are taken into consideration as fuzzy random variable due to their dual nature of fuzziness and randomness.

Index Terms—Fuzziness, Randomness, Advection, Contaminant migration, dispersion, retardation

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1. INTRODUCTION

The concept of fuzzy random variable, that extends the classical definition of random variable, was introduced by Feron [1] in 1976. Later on and sometimes independently, Kwakernaak [2, 3], Puri and Ralescu [4], Kruse and Meyer [5], Diamond and Kloeden [6], proposed other variants. In [7], Kratschmer surveyed all of these definitions and proposed a unified approach. In all of these papers, a fuzzy random variable is defined as a function that assigns a fuzzy subset to each possible output of a random experiment. The different definitions in the literature disagree on the measurability conditions imposed to this mapping, and in the properties of the output space, but all of them intend to model situations that combine fuzziness and randomness. Since the introduction of this concept, much effort has been devoted to the generalization of different probabilistic concepts and classical results to the case when outcomes of a random experiment are represented by fuzzy sets. Generalized definitions of descriptive parameters, useful as information summaries for probability distributions, can be divided into two groups. As an application of fuzzy random variable, in the field of quantification of uncertainty, migration of contaminants through soil layer just after the release may reach the biosphere to pollute or contaminate the environment is demonstrated. Monitoring or tracing of such contaminants in and around of any industry (nuclear or chemical) is one of the important program of environmental safety. The transport mechanisms, governing transport equation and the analytical solution of the transport equation are given for instantaneous and continuous sources [8-10]. The analytical solution play a major role in providing (1) an efficient model for controlled laboratory experiments and (2) a simple check to more complicated models that require numerical solutions. Moreover, calculation using simple analytical solution provides a conceptual knowledge of the effects of fuzzy randomness of sorption, transformation, advection and dispersion on the rate of subsurface transport. However, the physical processes like advection, dispersion and sorption being governed by the solution of the concentration of the contaminant that migrates through a soil layer. Fuzziness is due to imprecise measurement at any laboratory and randomness is due to the variability of the measured value across various laboratories. Randomness of the parameter is characterized by the standard probability distribution such as normal, lognormal, etc. and fuzziness is attributed by triangular or trapezoidal fuzzy number.

The fuzzy random parameters the model output provides a fuzzy random membership function of the fuzziness is obtained from expert's opinion. Fuzziness and randomness are complementary which can be brought under same umbrella by employing the great relationship between two named as fuzzy random variable (FRV). This new variable, FRV is a measurable function [11] from a probability space to the set of fuzzy variables [11]. FRVs can also be referred as random fuzzy sets or simply random sets [11].

The paper presents these FRVs in modelling contaminant (radionuclide or chemical) migration through soil layer. The paper addresses the distinction between fuzziness and randomness, preliminary concepts of probability, possibility and credibility required to understand the concept of, FRVs, various kinds of FRV model with a comparison among them and the fundamental concepts pertaining to the transport process of the migration of contaminant through a soil layer. Finally, the model describing the migration of the contaminant through soil layer presents a new direction of environmental safety analysis in presence of an admixture of variability and uncertainty due to randomness and fuzziness of the model parameters.

2. FUZZINESS AND RANDOMNESS – ARE THEY SAME?

In order to implement the concept of fuzzy random variable for assessing the uncertainty of any physical model, it is mandatory to know whether randomness and fuzziness are connotation or not. It has been already pointed out in section 1 that these two are complementary, so, fuzziness and randomness are not the same. Basically randomness to fuzziness is one kind of paradigm shift. Randomness addresses the variability of the uncertain variable whereas fuzziness describes the ignorance of the variable. Fuzziness can be reduced whereas randomness can't be reduced. Randomness is described by the probability distribution, whereas, fuzziness is represented by possibility distribution. Therefore, it can be envisaged that there exists a distributional difference as well as membership difference between fuzziness and randomness. Let us first focus on the distributional differences and demonstrate it by a simple example. A representative probability distribution, based on survey of radiation history of occupational workers in any nuclear plant, a probability of 0.94 can be assigned as zero overexposure, a 0.06 probability of one over exposure, and a 0.004 probability of two overexposures. In contrast, one can imagine that the representative possibility distribution of zero and one overexposure of occupational workers each has a high possibility of occurrence, and there is some possibility that the more than two occupational workers will be overexposed. Therefore, it can be concluded that, a probable event is always possible, while a possible event need not be probable. Zadeh (1978) called this heuristic connection between possibilities and probabilities the probability/possibility consistency principle [12]. This informal principle may be translated as: the degree of possibility of an event is greater than or equal to its degree of probability, which must be itself greater than or equal to its degree of necessity. Now consider the membership differences by accounting two groups of members of the public, one is within 5 km and the other is beyond 5 km of the site boundary of a radiochemical facility, and they are being classified as high dose and low dose category on the basis of the ingestion dose received. Ingestion dose received is due to the ingestion of contaminated food. There exists an uncertainty as to whether these groups are categorized perfectly or not in the sense that the dose received by the low dose category is very low and the dose received by the high category is moderately high. Low category is to be classified on the basis of ingestion dose received having membership of 0.8 whereas high being a removal category of 0.9. Assuming one or other has to be classified as traditional ingestion category is to be classified on the basis of probability and is known to have a probability of dose, which one should be accepted for safety assessment. In this situation an obvious conclusion is that fuzzy degrees are not the same as probability percentages. That is, grade of membership is different from probability of membership.

3. DEFINITIONS: PROBABILITY, POSSIBILITY, NECESSITY, CREDIBILITY AND BOREL SETS

In view of the difference between randomness and fuzziness, let us define formally the probability, possibility and credibility spaces. Basic features of these spaces are presented in Table 1 and following this we define them in the following way:

A. Probability

As indicated in Table 1, a probability space is defined as the 3-tuple ((Ω, A, Pr), where $\Omega = \{\omega_1, \omega_2, \omega_3, \ldots, \omega_N\}$ is a sample space. A is the σ-algebra of subsets of Ω and Pr, a probability measure on Ω, such that it satisfies:

$\Pr(\Omega) = 1$, $\Pr\{\phi\} = 0$, $0 \le \Pr\{A\} \le 1$ for any $A \in A$. For every countable sequence of mutually disjoint events $\{A_i\}$, i = 1, 2, ….

$$\Pr\left\{\bigcup_{i=1}^{\infty} A_i\right\} = \sum_{i=1}^{\infty} \Pr\{A_i\} \qquad (1)$$

ISSN 2229-5518

Table 1: Probability, Possibility and Credibility Spaces

Probability Space	Possibility Space	Credibility Space
(Ω, A, Pr) is a probability space	(Θ,$P(\Theta)$, Pos) is a possibility space	(Θ,$P(\Theta)$, Cr) is a credibility space
Ω: sample space	Θ: sample space	Θ:sample space
A: σ-algebra of subsets of Ω	$P(\Theta)$: power set of Θ	$P(\Theta)$: power set of Θ
Pr: probability measure on Ω	Pos: possibility measure on Θ	Cr: credibility measure on Θ

Probability measure satisfies the law of excluded middle (which requires that a proposition be either true or false), the law of contradiction (which requires that a proposition cannot be both true and false), and the law of truth conservation (which requires that the truth values of a proposition and its negation should sum to unity) [13].

B. Possibility

A possibility space from Table 1 is defined as the 3 tuple ((Θ,$P(\Theta)$, Pos), where $\Theta = \{\theta_1, \theta_2, \ldots, \theta_N\}$ is a sample space, P(Θ), also denoted as 2^Θ, is the power set of Θ, that is, the set of all subsets of Θ, and Pos is a possibility measure [6] defined on Θ. Pos{A}, the possibility that A will occur, satisfies: Pos{Θ} = 1, Pos{ϕ} = 0, $0 \le \text{Pos}\{A\} \le 1$, for any A in P($\Theta$)

$$Pos\left\{ \cup_i A_i \right\} = \sup_i Pos\left\{ A_i \right\} \qquad (2)$$

for any collection {A_i} in P(Θ). The heavy red line as shown in Fig. 1 represents the possibility of a fuzzy event characterized by $\zeta \ge x$, where ζ = (2,5,9) is a triangular fuzzy variable given by the mathematical form as

$$Pos\{\zeta \ge x\} = \begin{cases} 1, & x \le 5, \\ \dfrac{9-x}{9-5}, & 5 \le x \le 9, \\ 0, & x \ge 9 \end{cases} \qquad (3)$$

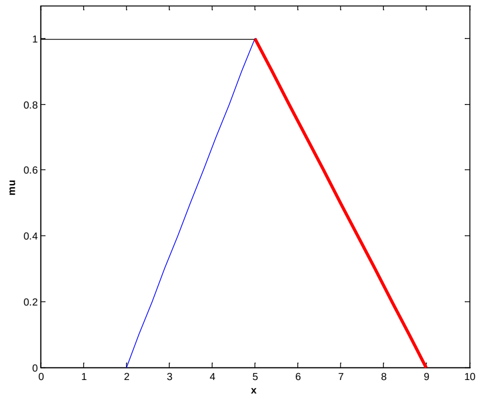

Fig. 1. Possibility that ζ is greater than x

It can be stated that the possibility of an event is determined by its most favourable case only, in contrast to the probability of an event, where all favourable cases are accumulated. By its very nature, the possibility measure is inconsistent with the law of excluded middle and the law of contradiction and does not satisfy the law of truth conservation [13].

C. Necessity

The necessity measure of a set A often is defined as the impossibility of the opposite set A^c [13]. Formally, let (Θ, P(Θ),Pos) be a possibility space, and A a set in P(Θ). Then the necessity measure of A is defined by

$$\text{Nec}\{A\} = 1 - \text{Pos}\{A^c\} \qquad (4)$$

Considering the triangular fuzzy variable ζ = (2,5,9), we can represent Nec{$\zeta \ge x$} by the mathematical equation, (The red line of Fig. 2 shows the necessity measure).

$$Nec\{\zeta \ge x\} = \begin{cases} 1, & x \le 2, \\ \dfrac{5-x}{5-2}, & 2 \le x \le 5, \\ 0, & x \ge 5 \end{cases} \qquad (5)$$

It can be noted from Fig. 2 that, Nec{$\zeta \ge x$}= 1- Pos{$\zeta < x$}.

ISSN 2229-5518

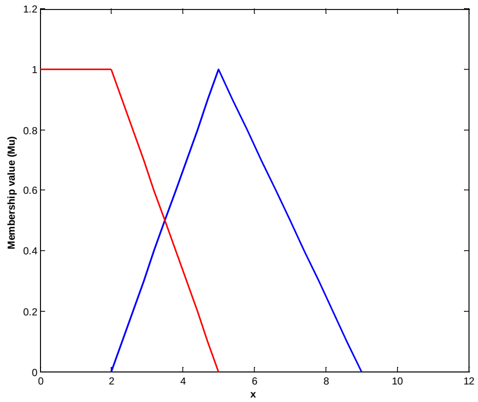

Fig. 2. Necessity that ζ is greater than or equal to x

D. Credibility

Given the limitations of the possibility measure, Liu and Liu (2002) suggested replacing it with what they termed as credibility measure [14]. The credibility measure takes the form

$$Cr\{X \le r\} = 0.5\,(Pos\{ X \le r\} + Nec\{ X \le r\}) \quad (6)$$

or, equivalently,

$$Cr\{X \le r\} = \frac{1}{2}\left(\sup_{t \le r} \mu_x(t) + 1 - \sup_{t > r} \mu_x(t)\right)$$

The set {Cr} on the power set *P* is called a credibility measure if it satisfies the following four axioms [15]

(1) Normality: $Cr\{\Theta\} = 1$

(2) Monotonicity: $Cr\{A\} \le Cr\{B\}$ whenever $A \subset B$

(3) Self-Duality: $Cr\{A\} + Cr\{A^c\} = 1$ for any event A

(4) Maximality: $Cr\{\cup_I A_i\} = \sup_i Cr\{A_i\}$ for any events $\{A_i\}$ with $\sup_i Cr\{A_i\} < 0.5$

The triplet (Θ, *P*(Θ), Cr) is called a credibility space. It can be stated that the credibility measure is a special type of non-additive measure with self-duality. In this context, a fuzzy event may fail even though its possibility achieves 1, and may hold even though its necessity is 0. However, the fuzzy event must hold if its credibility is 1 and fail if its credibility is 0. The mathematical representation of the credibility of ζ ≥ x can be written with the help of the given triangular fuzzy number (a, b, c) ≡ (2,5,9) as

$$Cr(\zeta \ge x) = \begin{cases} 1 & x \le a, \\ \dfrac{2b - a - x}{2(b-a)} & a \le x \le b, \\ \dfrac{c - x}{2(c-b)} & b \le x \le c, \\ 0 & x \ge c \end{cases} \quad (7)$$

The solid red line as shown in Fig. 3 represents the credibility value of the fuzzy event characterized by ζ ≥ x.

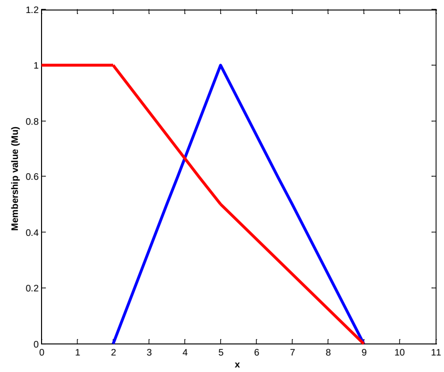

Fig.3. Credibility that ζ is greater than or equal to x

Concepts of fuzzy number and α-cuts can be found elsewhere in [16] and for that reason let us define Borel sets:

E. Borel sets

If *F* is a collection of subsets of the sample space, Ω, then *F* is said to be a σ-algebra if the following conditions hold: Ω ∈ *F*; if A ∈*F* then A^c ∈ *F*; and if A = $\bigcup_{i=1}^{\infty} A_i$ and A_i ∈ *F* for i ∈ I^+, then A∈ *F*. The Borel σ-algebra, *B* is the smallest σ-algebra that contains the set of all open intervals in **R,** the set of real numbers. Elements of *B* are called Borel sets and (**R,** *B*) is called Borel measurable space. An example can be given to clear the concept of Borel set. Let us consider the conflict case of consumption of wheat in a typical family in southern region of India. One group gave this figure as [100, 200] kg/yr and the other group said [150, 300] kg/yr. If at least one of these groups is correct, the consumption of wheat will fall within the union of two, i.e., [100,300]. But, if both the groups are correct, the consumption of wheat will fall in the intersection of their estimates, that is, the interval [150, 200]. Borel sets are used to describe these types of data.

4. FUZZY RANDOM MODELS

Three kind of fuzzy random variables are cited in the literature [2-5]. One is due to Kwakernaak (1978, 1979) [2-5], who coined the term "fuzzy random variable" and interpreted as FRVs as "random variables whose values are not real but fuzzy numbers". The second one is due to Puri and Ralescu (1986) [4], who regarded FRVs as random fuzzy sets. The third one is by Liu and Liu (2002, 2003). Accordingly, we have three kinds of fuzzy random variable models and we present all the FRV models as follows:

4.1 Kwakernaak FRV Model

In this model, a FRV is a mapping ζ: $\Omega \rightarrow F(\mathbf{R})$ such that for any $\alpha \in [0,1]$ and all $\omega \in \Omega$, the real valued mapping is as follows:

$\inf \zeta_\alpha : \Omega \rightarrow R,$ satisfying $\inf \zeta_\alpha(\omega) = \inf(\zeta(\omega))_\alpha$, and $\sup \zeta_\alpha : \Omega \rightarrow R,$ satisfying $\sup \zeta_\alpha(\omega) = \sup(\zeta(\omega))_\alpha$. These real valued mappings are real valued random variables, that is, Borel-measurable real-valued functions. These α-level constraints on ζ may be summarized as $\zeta_\alpha(\omega) = [\inf(\zeta(\omega))_\alpha, \sup(\zeta(\omega))_\alpha]$. In short, the Kwakernaak FRV takes the form of a mapping from Ω to the left and right hand side of the fuzzy target F(**R**), where the latter are real-valued random variables. If X be a FRV and Π_A is the collection of all A-measurable random variables of Ω, then the k^{th} moment of Kwakernaak FRV x, $E(x^k)$ is a fuzzy set on **R** with

$$\mu_{E(x^k)}(x) = \sup\{\mu_x(U) \mid U \in \Pi_A, EU^k = x\}, x \in R \quad (8)$$

The fuzzy variance of X is a fuzzy set $Var_k(x)$ on $[0,\infty)$ with

$$\mu_{var_k(x)}(\sigma^2) = \sup\{\mu_x(U) \mid U \in \Pi_A, D^2U = \sigma^2\}, \sigma^2 \in [0,\infty) \quad (9)$$

4.2 Puri and Ralescu FRV Model

Prior to present Puri and Ralescu (1986) FRV model [4], it is required to describe Banach space in short. Banach space is a normed linear space which is complete as a metric space. Banach spaces are used to extend the domain of FRVs from the real line to Euclidean n-space. Puri and Ralescu (1986) conceptualized a FRV as a fuzzification of a random set [4]. If (B, |.|) be a separable Banach space, *K*(B) be a nonempty compact subset of B, this model is addressed a FRV as a mapping ζ: $\Omega \rightarrow F(B)$ such that for any $\alpha \in [0,1]$ the set-valued mapping $\zeta_\alpha : \Omega \rightarrow K(B)$ (with $\zeta_\alpha(\omega) = (\zeta(\omega))_\alpha$ for all $\omega \in \Omega$) is a compact random set, that is, it is Borel-measurable with the Borel σ-field generated by the topology associated with the Hausdorff metric on *K*(B) [16]

$$d_H(P,Q) = \max\{\sup_{p\in P} \inf_{q\in Q} |p-q|, \sup_{q\in Q} \inf_{p\in P} |p-q|\}$$

If P and Q are bounded, then the Hausdorff metric becomes

$$d_H(P,Q) = \max\{|\inf p - \inf q|, \ |\sup p - \sup q|\} \quad (11)$$

4.3 Expectation value of a Puri and Ralescu FRV

It is known that a FRV ξ is said to be an integrably bounded FRV associated with the probability space (Ω, *A*, P) if an only if $\|\xi_0\| \in L^1(\Omega, A, P)$, where, for the function f,

$$L^1(\Omega, A, P) = \{f \mid f : \Omega \rightarrow R, A - \text{measurable}, \ \int |f|^1 dP < \infty\}$$

Now, given the probability space (Ω, *A*, P), ξ an integrably bounded FRV associated with (Ω, *A*, P), and S(F) a nonempty bounded set with respect to the L^1(P)-norm, the expected value of ξ is the unique fuzzy set $\tilde{E}(\xi \mid P)$ of $\mathbf{R^n}$ such that $(\tilde{E}(\xi \mid P))_\alpha = \int_\Omega \xi_\alpha dP$ for all $\alpha \in [0,1]$, where $\int_\Omega \xi_\alpha dP = \{\int_\Omega f \ dP \mid f \in S(\xi_\alpha)\}$ is the Aumann integral [17] of ξ_α with respect to P. Puri and Ralescu defined the expected value (EV) of a FRV as a generalization of the EV. Operationally, when a fuzzy random variable ξ: $\Omega \rightarrow F(\mathbf{R})$ is integrably bounded, the EV of ξ is unique and for all $\alpha \in [0,1]$, is given by the compact interval $[E(\inf \xi_\alpha), E(\sup \xi_\alpha)]$.

4.4 Variance of a Puri and Ralescu FRV

The variance should be used to measure the spread or dispersion of the FRV around its EV [18]. Accordingly, scalar variance of a Puri and Ralescu FRV is defined as

$$V(\overline{X}) = \frac{1}{2}\int_0^1 [V(\underline{X}_\alpha) + V(\overline{X}_\alpha)] \, d\alpha \quad (12)$$

4.5 The Liu and Liu FRV Model

Liu and Liu(2002, 2003) [19] expressed concern that both the Kwakernaak and Puri and Ralescu
RV models were based on the possibility measure, and, as such, did not obey the law of truth conversation and were inconsistent with the law of excluded middle and the law of contradiction. To overcome these perceived shortcomings, they based their FRV on the credibility measure [20],

$$Cr\{A\} = 0.5(Pos\{A\} + 1 - Pos\{A^c\}) \quad (13)$$

which they contended plays the role of probability measure more appropriately than either the possibility and necessity measures. Finally, their FRV model incorporated a scalar, rather than a fuzzy, expected value, since they viewed the latter as problematic from an implementation perspective. On the basis of credibility measure, Liu (2006) defines a fuzzy random variable as a function ξ from a probability space (Ω,)
A, Pr) to the set of fuzzy variables such that $Cr\{\xi(\omega) \in B\}$ is a measurable function of ω for any Borel set *B* of **R** [19].

4.6 Expectation value of the Liu and Liu FRV

If ξ is fuzzy random variable defined on the probability space (Ω, *A*, Pr), then the expected value of ξ as per Liu and Liu (2006) is defined as [19, 20]

$$E[\xi] = \int_0^{\infty} Cr\{\xi \geq x\}\, dx - \int_{-\infty}^{0} Cr\{\xi \leq x\}\, dx$$

provided that at least one of the two integrals is finite. Alternatively, an expected value of ξ can be conceptualized as

$$E[\xi] = \int_{\Omega} [\int_0^{\infty} Cr\{\xi(\omega) \geq r\}\, dr - \int_{-\infty}^{0} Cr\{\xi(\omega) \leq r\}\, dr] Pr(d\omega) \quad (15)$$

provided that at least one of the two integrals is finite and in the event that ξ is a nonnegative fuzzy random variable, expectation of ξ can be written as

$$E[\xi] = \int_{\Omega} \int_0^{\infty} Cr\{\xi(\omega) \geq r\} dr \;\; Pr(d\omega) \qquad (16)$$

According to the definition of the expected value of fuzzy variable, ξ as given in Eq. (14), the equipossible fuzzy variable on [a, b] has an expected value (a + b)/2. The triangular fuzzy variable (a, b, c) has an expected value (a + 2b+ c)/4; this can be derived as follows:

$$E[\xi] = b + \frac{1}{2}\int_b^c \frac{x-c}{b-c} dx - \frac{1}{2}\int_a^b \frac{x-a}{b-a} dx$$
$$= b + \frac{c-b}{4} + \frac{a-b}{4} = \frac{a+2b+c}{4}$$

In a similar way, the trapezoidal fuzzy variable (a, b, c, d) has an expected value of (a + b + c + d)/4. As an example, a fuzzy variable, ξ is called exponentially distributed if it has an exponential membership function

$\mu(x) = 2\left(1 + \exp\left(\frac{\pi\, x}{\sqrt{6}\; m}\right)\right)^{-1}$, $x \geq 0, m \geq 0$; where, m is the parameter of the exponential membership function; The expected value of the FRV, ξ is $\sqrt{6}\, m \ln 2/\pi$. Further, if ξ is a normal fuzzy variable with normal membership function

$\mu(x) = 2(1 + \exp(\frac{\pi|x-e|}{\sqrt{6}\sigma}))^{-1}, x \in R, \sigma > 0,$ then the expected value of ξ is e.

4.7 Variance of the Liu and Liu FRV

If ξ is a FRV with finite expected value E[ξ], the variance of ξ according to Liu and Liu (2003), is defined as the expected value of the FRV $(\xi - E[\xi])^2$ [19, 20]. Therefore, we can write

$$Var[\xi] = E[(\xi - E[\xi])^2] \qquad (17)$$

If ξ is the fuzzy normal variable, then the variance of ξ is

(14)

$$Var[\xi] = \int_e^{\infty} (1 + \exp(\frac{\pi\sqrt{r}}{\sqrt{6}\sigma})^{-1}\, dr = \sigma^2$$

5. FUZZY RANDOM CONTAMINANT MIGRATION MODEL

5.1 Mathematical formulation

The model in one dimension is formulated on the basis of the following assumptions:

- The porous medium is assumed as homogeneous, isotropic and saturated.
- The flow is steady so that Darcy's law holds good

The flow is described by the seepage velocity, which transports the dissolved substance by advection. One-dimensional advection-dispersion equation for a reactive contaminant with first order contaminant degradation including radioactive decay is given by [21, 22]

$$\tilde{R}\frac{\partial \tilde{C}}{\partial t} = \tilde{D}_x \frac{\partial^2 \tilde{C}}{\partial x^2} - \tilde{v}_x \frac{\partial \tilde{C}}{\partial x} - k\tilde{C} - \lambda\tilde{C} \quad (18)$$

where $\tilde{D}_x$ = FRV, groundwater dispersion coefficient (m^2/day); $\tilde{C}$ = FRV, contaminant concentration in the aqueous phase (mg/l); x = distance (within the soil layer); $\tilde{R}$ = FRV, the retardation factor = $[1 + (\rho_b K_d / \tilde{n})]$; for non-reactive solutes, the retardation factor is a crisp number and R = 1; ρ_b = the bulk mass density of the porous medium; $\tilde{K}_d$ = FRV, the distribution coefficient; $\tilde{n}$ = FRV, the effective porosity; $\tilde{v}_x$ = FRV, pore water velocity (m/day) (seepage velocity); k = first order rate constant for contaminant degradation (day^{-1}) and it is a crisp number; λ = radioactive decay constant of the specific nuclide of interest (day^{-1}), t = time of observation (day). A number of analytical solutions of Eq. (18) on the basis of its integration with the specific boundary conditions for the system of interest provide the transport of contaminants in groundwater systems. The fuzzy random model presented in this paper is a pulsed model because a pulse of contaminant, such as a leaking underground storage tank or the site of an accidental hazardous chemical materials spill is used as an input.

ISSN 2229-5518

Assumptions used to obtain the analytical solution of Eq. (18) are:

- The tracer is ideal, with constant density and viscosity;
- The fluid is incompressible;
  - The medium is homogeneous and isotropic;
  - Only saturated flow is considered.

5.2 Pulse model

A tracer pulse (instantaneous input) at x = 0 is used with zero background concentration. As the contaminant moves downstream with the seepage velocity, v_x in the +x direction, it spreads out which is mathematically formulated by the solution of Eq. (18) as

$$\tilde{C}(x,t) = \frac{Q}{\sqrt{4\pi \tilde{D}_x^{'} t}} \exp\left[-\frac{(x-\tilde{v}_x^{'} t)^2}{4\pi \tilde{D}_x^{'} t}\right] \times \eta \times \zeta$$

$$where,\ \eta = e^{-k^{'} t} \text{ and } \zeta = \exp(-\lambda^{'} t) \qquad (19)$$

where Q = the quantity of contaminant spilled per cross-sectional area (g/m²), $\tilde{D}_x^{'} = \tilde{D}_x / \tilde{R}$ (m²/day), $\tilde{v}_x^{'} = \tilde{v}_x / \tilde{R}$ (m/day), $k^{'} = k/\tilde{R}$, the first order loss coefficient (day^{-1}), $\lambda^{'} = \lambda / \tilde{R}$ (day^{-1}) and $\tilde{R}$ = the retardation factor. All the symbols marked with tilde (~) sign in Eq. (19) are fuzzy random variable (FRV). Retarded fronts can be derived from conservative fronts by adjusting the value of $\tilde{D}_x^{'} = \tilde{D}_x / \tilde{R}$ and $\tilde{v}_x^{'} = \tilde{v}_x / \tilde{R}$ in one dimension. Larger values of $\tilde{D}_x$ tend to spread out the fronts, while large values of the retardation factor tend to slow the velocity of the centre of the tracer (C/C_0 = 0.5) and reduce $\tilde{D}_x$ by a factor of $1/\tilde{R}$.

6. RESULTS AND DISCUSSIONS

The parameters D_x, v_x, and n of the contaminant migration model (Eq. (19) and Eq. (20)) are fuzzy random variable. On the basis of the experimental values measured at any standard laboratory, the randomness fuzziness [23, 24] property of the dispersion coefficient, D_x is taken into account as a fuzzy normal random variable, the mean of which is a triangular fuzzy variable and standard deviation is the 10% of the most likely value of the triangular fuzzy number. Following the same strategy, porosity of the soil, 'n' is represented here as a fuzzy uniform variable, for which both the lower and upper limits are a triangular fuzzy number. The randomness of the fuzzy random parameter seepage velocity or pore water velocity, v_x, is represented by a lognormal distribution, specified by the fuzzy geometric mean and crisp geometric standard deviation. In order to sample this, geometric mean and geometric standard deviation are transformed into the corresponding arithmetic mean and arithmetic standard deviation. Since the geometric mean is a fuzzy number, fuzziness of the corresponding arithmetic mean (AM) is formed as triangular fuzzy number by subtracting the arithmetic standard deviation (ASD) from the AM and by adding the same with ASD. The density ρ of the soil, the distribution coefficient, K_d and the degradation constant, k are represented as crisp variable. Radioactive decay constant is obviously a crisp number. Monte Carlo simulation technique is used to generate the sample values of random part of the FRV whereas alpha cut method is used to generate the compact interval of the fuzziness part of FRV. Puri and Ralescu FRV model is used to represent the FRVs of the present model and accordingly, from the point of simplicity of computation the corresponding mathematical structures of these fuzzy random variables are as follows:

$$\tilde{D}_x \sim N([\mu_l, \mu_m, \mu_r], \sigma),\ \tilde{v}_x \sim Ln([AM = \tilde{\mu}, ASD = \sigma]),$$
$$\tilde{\mu} = Ln([\mu_L, \mu_M, \mu_R], \sigma),\quad n \sim U([l_1, u_1],\ [l_2, u_2])$$

Algorithm followed to generate the sample values of these FRVs are as follows:

Step 1. Generate sample values of D_x: Alpha cut representation of the fuzzy number μ, is first generated for each α varying from 0 to 1 with an increment of 0.1. Each such alpha cut values of μ is a compact interval such as $[\mu_{LB}^{\alpha}, \mu_{UB}^{\alpha}]$. Now lower and upper bound of alpha cut representation of D_x are generated by traditional sampling techniques of a normal distribution, lower bound with $N(\mu_{LB}^{\alpha}, \sigma)$ and the other one with $N(\mu_{UB}^{\alpha}, \sigma)$.

Step 2. Generate sample values of v_x: Alpha cut representation of the AM (triangular fuzzy number) of v_x is constructed for each $\alpha \in [0,1]$ with an increment of 0.1. Finally, using the traditional Monte Carlo sampling of a lognormal distribution with $[AM_{LB}^{\alpha}, ASD]$ and with $[AM_{UB}^{\alpha}, ASD]$, lower and upper bounds of sample values of pore water velocity are generated.

Step 3. Generate sample values of n: Alpha cut representation of the lower and upper limits of the uniform distribution (both are triangular fuzzy number) of n are constructed for each $\alpha \in [0,1]$ with an increment of 0.1. So, here we have the structure as n ~ U([LB_1^α, UB_1^α], [LB_2^α, UB_2^α]. All these bound values being positive number, sample values of the lower and upper bound of n are generated by traditional Monte Carlo sampling of an uniform distribution with U([LB_1^α, LB_2^α] and with U[UB_1^α, UB_2^α] respectively.

Step 4. Generate sample values of R: Lower and upper bound of each alpha cut level of the derived parameter, retardation factor R are constructed by using the expression of R as given by the expression$[1+(\rho_b K_d / \tilde{n})]$.

Step 5. Compute C: The lower and upper bounds of fuzzy random distribution of the leachate concentration, C are generated by using these sample values in Eq. (19). Two sets of concentration matrices, one for varying downstream distance and fixed time, and the other for fixed downstream distance and varying time are generated. The lower and upper bounds of each set of concentration matrix is populated with all the alpha cuts (i.e., α= 0 to 1 with a step size of 0.1) as columns and sample size of the random distributions as rows. As 1000 sample values of each of the FRV were generated, number of rows of concentration matrix are 1000. Therefore, the dimension of the each lower and upper bounds of the concentration matrix for for each such set is (1000 x 11). Length of downstream distance array is used as 50 (x = 1 m to 500 m with a step size of 10 m) and that of time array as 10 (t = 1 y to 10 y with a step size of 1 y). Considering the distance and time array our new dimension of lower and upper bounds of concentration matrix become (1000 x 11 x 50) for first set (varying distance but fixed time) and (1000 x 11 x 10) for the second set (varying time but fixed downstream distance). Therefore, a mathematical structure of concentration matrix can be written as

$$\{[\tilde{C}_{LB}{}^{\alpha}]_{(1000\times 11\times 50)},[\tilde{C}_{UB}{}^{\alpha}]_{(1000\times 11\times 50)}\}_{(ttime)}$$

$$\{[\tilde{C}_{LB}{}^{\alpha}]_{(1000\times 11\times 10)},[\tilde{C}_{UB}{}^{\alpha}]_{(1000\times 11\times 10)}\}_{(dis\tan ce)}$$

Dimension of these matrices tell us that a substantial amount of post processing is required to understand the role of fuzzy random variable in modelling contaminant migration through a soil layer. Results of the implementation of fuzzy random in modelling migration of contaminant through soil layer via post processing and the corresponding discussions are presented in section 7.

7. RESULTS AND DISCUSSIONS

Input data set used for executing the fuzzy random model of contaminant migration through soil layer is Dispersion coefficient, $\tilde{D}_x$ (m^2/day): Fuzzy normal: N(Fuzzy (TFN) μ, Crisp σ) = N ([0.6, 0.8, 1.0], 0.08), Porosity of soil, n: Fuzzy uniform, U([0.2, 0.4, 0.6], [0.7, 0.8, 0.9]), Seepage (pore water) velocity, $\tilde{v}_x$:Fuzzy lognormal, Ln(GM_{TFN}, GSD_{TFN}), TFN: Triangular fuzzy number, Ln(<2,5,7>, <3,6,8>), Distribution coefficient, K_d (m^3/kg) = 2.8, Density of the soil, ρ (kg/m^3) = 1.4, Degradation coefficient, k (day^{-1}) = 0.0004, Radionuclide considered is ^{137}Cs, Radioactive decay constant, λ (day^{-1}) = 30 y, Source term, Q = 1740 g/m^2, Downstream distances are: 1 m to 500 m with a step size of 10 m and time of observations taken into consideration are: 1 to 10 y with a time step of 1 y.

Results of set 1:

In this set, computation of fuzzy random concentration of contaminant is carried out at a fixed time of 1y (365 days) and at various downstream distances ranging from 1 m to 500 m with a step size of 10 m. Randomness of the fuzzy random parameter longitudinal dispersion coefficient is addressed by the cumulative distribution function at various alpha cuts of its fuzziness as shown in Fig. 4, whereas its fuzziness is addressed by the triangular membership function as presented in Fig. 5. The randomness of the fuzzy random parameter seepage (pore water) velocity is shown in Fig.6 in the form of left and right CDF at various alpha cuts. Membership function corresponding to mean value, 5th and 95th percentiles of the randomness of the derived fuzzy random parameter, the retardation factor is presented in Fig. 7. It can be easily make a comment on Fig. 7 that derived fuzziness cannot be a L-R fuzzy number.

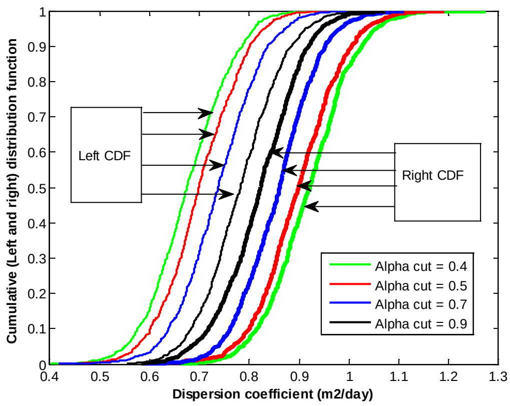

Fig. 4 Left and right CDF (randomness) of dispersion coefficient at various

alpha cut value of its fuzziness

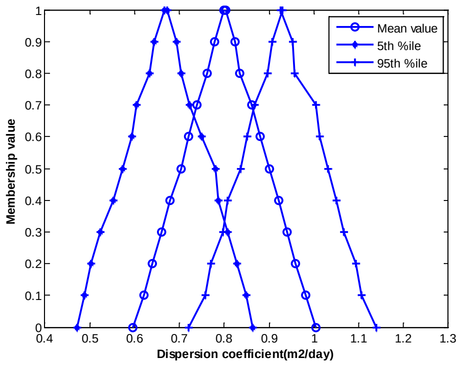

Fig. 5 Triangular membership function of dispersion coefficient corresponding to its mean, 5th and 95th percentile of CDF

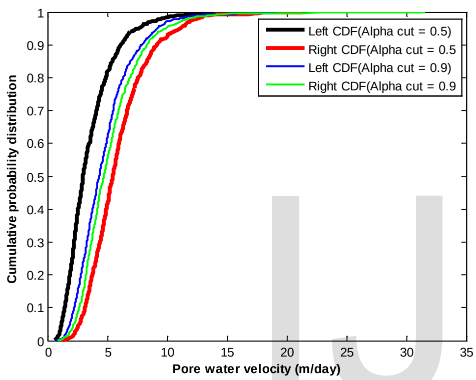

Fig. 6 Left and right CDF (randomness) of pore water velocity

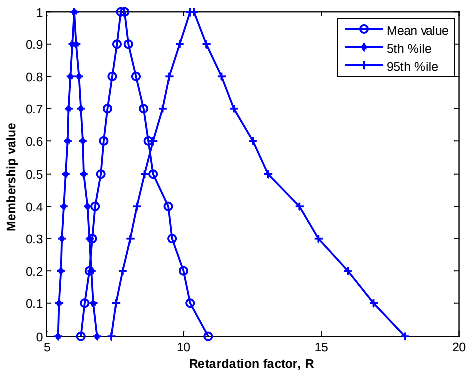

Fig. 7 Triangular membership function of mean value, 5th and 95th percentiles of Retardation factor

In a similar way, the fuzzy random parameter, porosity of the soil also can be also represented but due to its similar structural behaviour of fuzziness as well as randomness it has not been shown just simply to avoid the repetition of the same figure. Substitution of all these fuzzy random parameters into Eq. (19) with the other specified fixed input parameters including the source term 'Q' results the fuzzy random concentration of the contaminant at the target set points. The concentration of the contaminant now being a fuzzy random variable has two modes: fuzziness and randomness. At any alpha cut level it will have randomness which will be governed by a pulse-shaped Gaussian like distribution. Therefore, if the mean value of the randomness of the contaminant concentration is estimated for each downstream distance at any alpha cut level, one can have the profile of the mean concentration of the contaminant over downstream distances at any alpha cut level and the same has been presented at alpha cut of 0.5 in Fig. 7. Subsequently the lower bound of the pulsed mean concentration profile over different downstream distances for other alpha cuts (0.1, 0.2, 0.3 and 0.7) is presented in Fig. 8. It is now evident clearly from the Fig. 7 that the lower and upper bound of the mean concentration profile (alpha cut value) over distance is shaped as Gaussian like pulse. The trailing part of the lower and upper bound of the pulse (Fig. 7) is not exactly as trailing part of pure Gaussian shaped pulse and the obvious reason is due to the fuzziness of the concentration as well. Continuing the same discussion, it can be stated from Fig. 7 that 0.5 alpha cut value of the peak of the pulse of the mean concentration results at two different downstream distances (lower peak occurred at 100 m and the upper peak occurred at 240 m).

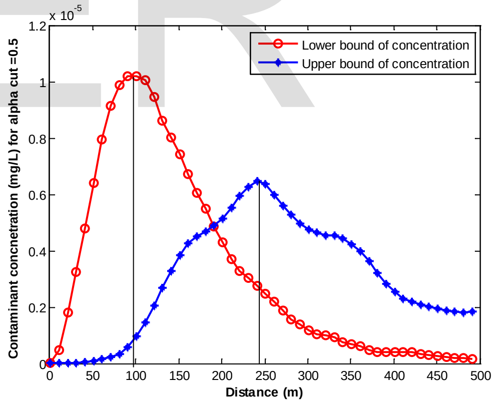

Fig. 7 Lower and upper bound profile of Mean concentration of contaminant at alpha cut of 0.5

It can be also stated that the alpha cuts are nothing but the possibility value of the fuzziness part of the mean concentration. Therefore, it can be also interpreted as the possibility of the maximum value of the mean concentration at distance 100 m and at 240 m is 0.5.

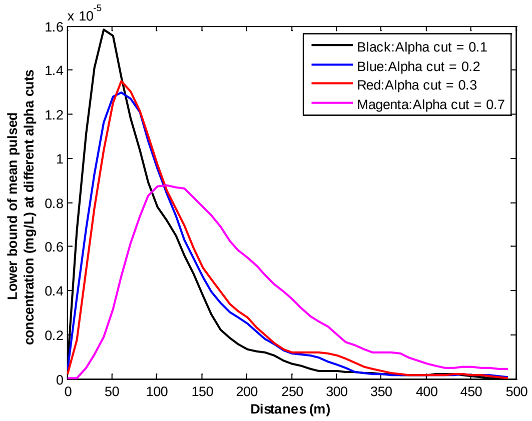

Fig. 8 Lower bound profile of Mean concentration of contaminant for different alpha cuts

It is obvious that fuzzy random concentration of the contaminant will have its fuzziness at every downstream distance. Accordingly, the fuzziness of the mean concentration at 91.0 m downstream distance is shown in Fig. 9.

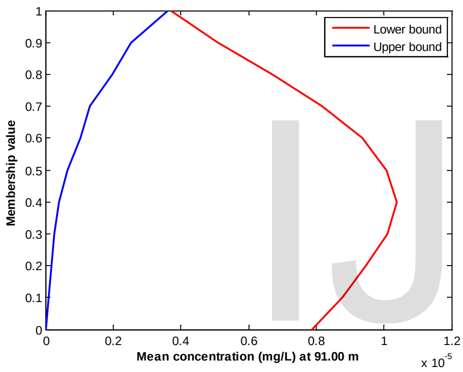

Fig. 9 Membership function of Mean concentration at 91.0 m

In order to know the variation of the fuzziness of mean concentration of the contaminant over different downstream distances, membership functions of the mean concentration at downstream distances 391 m, 441 m and 491 m respectively are shown in Fig. 10.

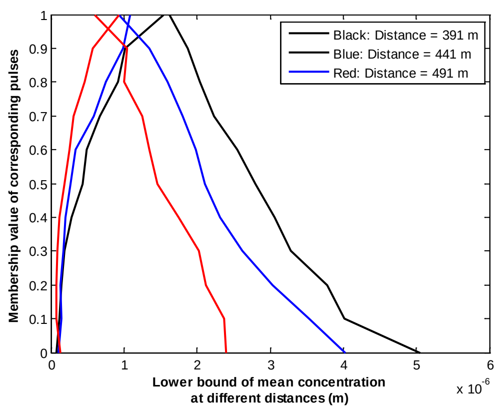

Fig. 10 Membership function of mean concentration at different downstream distance

Results of set 2:

In this set, computation of fuzzy random concentration of contaminant is carried out at a fixed downstream distance, x = 150 m and at various time ranged from 1 y to 10 y with a time step of 1y. Time profile of the mean value of fuzzy random concentration at alpha cut of 0.5 is shown in Fig. 11.

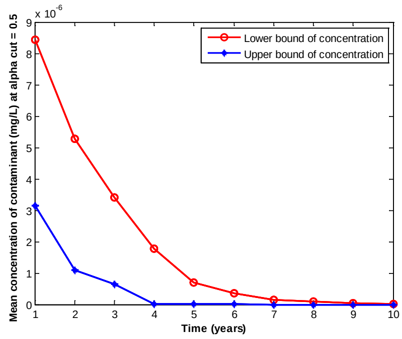

Fig. 11 Time profile of mean value of fuzzy random concentration at alpha cut of 0.5

Lower bound of the temporal profile of the mean value of fuzzy random concentrations at alpha cut of 0, 0.1 and 0.2 are shown in Fig. 12

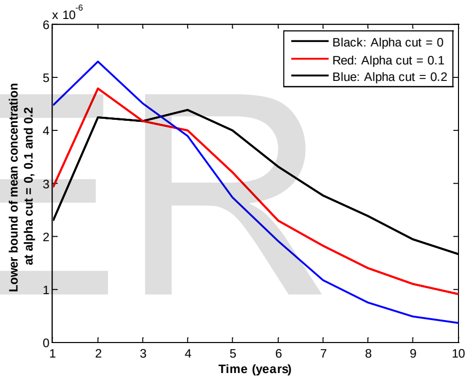

Fig. 12 Time profile of lower bound of mean value of fuzzy random concentration at alpha cut of 0, 0.1 and 0.2

In order to investigate the fuzziness of the mean value of fuzzy random concentration at a specific downstream distance (150 m) and at different time membership function of the same at time 2 y, 3y and 4 y are presented in Fig. 13. It is clearly understood from Fig. 13 that fuzziness of the mean concentration at all the time is not pure triangular membership function.

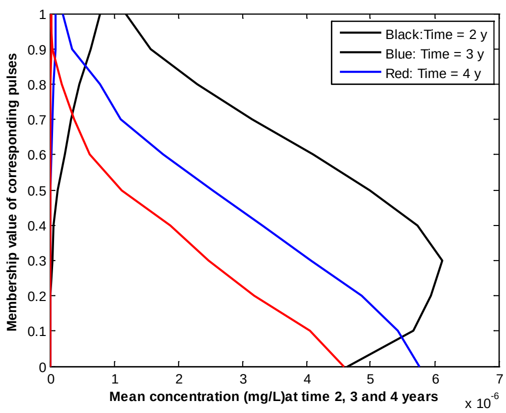

Fig. 13 Fuzziness of mean value of fuzzy random concentration at time 2 y, 3y and 4 y

REFERENCES

[1]. R. Feron, 1976. Ensembles aleatoires flous, C.R. Acad. Sci. Paris Ser. A 282 903-906.

[2]. Kwakernaak, H., 1978. Fuzzy random variables – I. definitions and theorems, Information Sciences, 15(1), 1-29.

[3]. Kwakernaak, H., 1979. Fuzzy random variables – II. Algorithms and examples for the discreet case, Information Sciences, 17(3), 253-278.

[4]. Puri, M.L., Ralescu, D.A., 1986. Fuzzy random variables, Journal of Mathematical Analysis and Applications, 114, 409-422.

[5]. R. Kruse, K.D. Meyer, 1987. Statistics with vague data. D. Reidel Publishing Company.

[6]. P. Diamond, P. Kloeden, 1994. Metric Spaces of Fuzzy Sets, World Scientific, Singapur.

[7]. V. Kratschmer, 2001. A unified approach to fuzzy random variables, Fuzzy Sets and Systems 123 1-9.

[8]. Rowe, R.K. and Booker, J.R., 1985. 1-D pollutant migration in soils of finite depth. ASCE Journal of Geotechnical engineering, 111 (4), 479-499.

[9]. Sharma, H.D. and Lewis, S.P., 1994. Waste containment systems, waste stabilisation and landfills: design and evaluation. Wiley, New York.

[10]. Bedient, P.N., Rifai, H.S., Newell, J.C., 1994. Ground water contamination transport and remediation. Prentice Hall, Englewood Cliffs, NJ.

[11]. Gonazalez-Rodriguez, G., Coulbi, A., Gil, M.A., 2006. A fuzzy representation of random variables: An operational tool in exploratory analysis and hypothesis testing, Intl. J. of Computational Statistical Data Analysis, 51, 163-176.

[12]. Zadeh, L.A., 1968. Probability measures of fuzzy events, Journal of Mathematical Analysis and Applications, 23, 421-427.

[13]. Dubois, D., and Prade, J., 1980. Fuzzy Sets and Systems: Theory and Applications. In: Mathematics in Science and Engineering, 144, Academic Press, New York.

[14]. Liu, B., and Liu, Y. K., 2002. Expected value of fuzzy variable and fuzzy expected value models, IEEE Transaction on Fuzzy Systems, vol. 10, pp. 445-450.

[15]. Liu, B., 2005. Foundations of Uncertainty Theory, Department of Mathematical Sciences, Tsinghua University.

[16]. Klir, G. J., and Yuan, B., 1995. Fuzzy sets and Fuzzy Logic: Theory and Applications, Prentice Hall, PTR, Upper Saddle River, NJ.

[17]. Anuman, R.J., 1965. Integrals of set-valued functions, J. Math. Anal. Appl., 12, 1-12.

[18]. Feng, Y., Hu, L. And Shu, H., 2001. The Variance and Covariance of Fuzzy Random Variables and their Applications, Fuzzy Sets and Systems, 120, 487-497.

[19]. Liu, Y.K. and Liu, B., 2003. Fuzzy Random Variables: A Scalar Expected Value Operator, Fuzzy Optimization and Decision Making, 2(2), 143-160.

[20]. Liu, B., 2006. A survey of credibility theory, Fuzzy Optimization and Decision Making, 5(4), 387-408.

[21]. Shackelford, C.D., 1993. In: Daniel, D.E. (Ed.). Geotechnical practice for waste disposal. Chapman & Hall, London.

[22]. Watts, R.J., 1998. Hazardous waste: sources, pathways, receptors. Wiley, New York.

[23]. Moller, B. And Beer, M., 2004. Fuzzy Randomness: Uncertainty in Civil Engineering and Computational Mechanics, Springer, Berlin Heidelberg, New York.

[24]. Liu, B., 2007, Uncertainty Theory, 2nd ed., Springer-Verlag, Berlin.